ON HYPERBOLIC BESSEL PROCESSES AND BEYOND

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Abstract. We investigate distributions of hyperbolic Bessel processes. We find links between the hyperbolic cosinus of the hyperbolic Bessel processes and the functionals of geometric Brownian motion. We present an explicit formula of Laplace transform of hyperbolic cosinus of hyperbolic Bessel processes and some interesting different probabilistic representations of this Laplace transform. We express the one-dimensional distribution of hyperbolic Bessel process in terms of other, known and independent processes. We present some applications including a new proof of Bougerol's identity and it's generalization. We characterize the distribution of the process being hyperbolic sinus of hyperbolic Bessel processes.

1. Introduction

The important role which functionals of Brownian motion play in many fields of mathematics (as mathematical finance, diffusion processes in random environment, probabilistic studies related to hyperbolic spaces etc.) is motivation to study the wide class of different diffusion processes connected to those functionals somehow (see for example [7], [5], [9], [14] and [15]). In this work we consider the special class of such diffusions: Brownian motion with a special stochastic drift. It is well known in literature that processes like Ornstein-Uhlenbeck, Bessel processes or radial Ornstein-Uhlenbeck are examples of Brownian motion with stochastic drift connected to special functions: parabolic cylinder, Bessel or Kummer functions (see [4]). In this work we consider another interesting class of such diffusions - hyperbolic Bessel process which are connected to Legendre functions. We establish the distribution of a hyperbolic Bessel process. Borodin presented the computation of the transition density function for hyperbolic Bessel process (see formulas (5.4) and (5.5) in [4]). The method used by Borodin relied on the connection between the Laplace transform of transition density function and two increasing and decreasing solutions of some ordinary differential equation of the second order (for the more details of this method see also [5]). Gruet established the transition density functions of hyperbolic processes by the planar geometry method (see [11], [10]). However,

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the results obtained by both authors are technically very complicated. The role of the hyperbolic Bessel processes in the world of Brownian motion functionals was mentioned also by Byczkowski, Małecki and Ryznar (see [6]).

Our research of hyperbolic Bessel process is completely different (when compared to the mentioned) and purely probabilistic. We present results for hyperbolic Bessel processes with the index $\alpha \geq -1/2$ and for the fixed time. We find the link between a hyperbolic cosinus of hyperbolic Bessel process and functionals of geometric Brownian motion. We present different probabilistic representations of Laplace transform of a hyperbolic cosinus of hyperbolic Bessel process R. We also give probabilistic representations of density of $\cosh R_t$. The link between hyperbolic Bessel process and functionals of geometric Brownian motion enable us to establish simple explicit form of Laplace transform of the vector $(e^{B_t+kt}, \int_0^t e^{2(B_u+ku)}du)$ for a standard Brownian motion B and an nonnegative integer k. These results are obtained for fixed t (not for time being stochastic) and that they can be effectively used in numerical computations. We express the distribution of hyperbolic Bessel process R in terms of squared Bessel process X and vector $(B^{(\mu)}, A^{(\mu)})$ of Brownian motion with drift and the integral of geometric Brownian motion, which is independent of X. The joint distribution of $(B^{(\mu)}, A^{(\mu)})$ is known in literature and is expressed via the density function (e.g. see [14]). As an interesting example of applications we find a new simple proof of the Bougerol's identity. We also characterize the distribution of $\sinh(R)$ for R being hyperbolic Bessel process. It is interesting that the distribution of such process is deduced from the stochastic process which is in a sense a generalization of a squared Bessel process. At the end, we outline the case of time being exponential random variable independent of a Brownian motion driving the hyperbolic Bessel process. In this case many explicit and computable results are known in literature (see for example [5], [14] or [15]).

2. Hyperbolic Bessel processes with fixed time

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_\infty$. We define a hyperbolic Bessel process R with index $\alpha \in [-1; \infty)$ as a diffusion, starting from a nonnegative x, given by

$$R_t = x + B_t + \big(\alpha + \frac{1}{2}\big)\int_0^t \coth R_u du. \tag{1}$$

There exits a unique solution to (1) defined up to explosion time and behavior of R depends on the index α. For $\alpha \neq -1/2$, the point 0 is an isolated singular point of (1). For $\alpha > -\frac{1}{2}$, a solution of (1) is strictly positive for $x > 0$ and is unique up to $T_a = \inf\{t \geq 0 : \ X_t = a\}$ for every $a > x \geq 0$ (see [8, Theorem 2.16]). In particular, for $\alpha > -\frac{1}{2}$ and $x \geq 0$ there exits a unique solution defined up to the explosion time. For $\alpha = -\frac{1}{2}$, a process R satisfying (1) is a Brownian motion starting from x. For $\alpha \in [-1; -\frac{1}{2})$, a solution of (1) is strictly positive for $x > 0$ and negative for $x = 0$. It is unique up to $T_a = \inf\{t \geq 0 : \ X_t = a\}$ for every $a > x \geq 0$ (see [8, Theorem 2.17]). In particular, for $\alpha \in [-1; -\frac{1}{2})$ and $x > 0$ there exits a unique positive solution defined up to the explosion time.

Recall that Gruet ([11]) defines a hyperbolic Bessel process with index $\alpha > -1/2$ as a nonnegative diffusion R with generator

$$\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2} + \big(\alpha + \frac{1}{2}\big)\coth(x)\frac{d}{dx}. \tag{2}$$

In the case of $-1/2 < \alpha < 0$ the definition is completed by requirement that 0 is an instantaneously reflecting point. It is also assumed that a starting point is nonnegative. The similar definition of hyperbolic Bessel process can be found in Borodin [4]. In [11], by the identification of the Green function, it is stated that a reasonable candidate for hyperbolic Bessel process with $\alpha = -1/2$ is a reflected Brownian motion (see the end of Section 3). We define a hyperbolic Bessel process with $\alpha = -1/2$ as a Brownian motion instead of the reflected one, since a Brownian motion is a solution of the (1). Such an extension of definition of hyperbolic Bessel process by a Brownian motion allows to recover the interesting results about the distribution of $\int_0^t e^{2B_u} du$ (including the Bougerol's identity).

To start the discussion about the properties of hyperbolic Bessel process we define a process θ being the solution of the SDE

$$d\theta_t = \sqrt{|\theta_t^2 - 1|} dB_t + (\alpha+1)|\theta_t| dt, \tag{3}$$

where $\alpha \geq -1$, and such that $\theta_0 = x \geq 1$. Observe that the diffusion coefficient $\sigma(x) = \sqrt{|x^2-1|}$ is locally Lipschitz. Moreover, for $b(x) = (\alpha+1)|x|$ and for $x \in \mathbb{R}$,

$$|\sigma(x)| + |b(x)| \leq (2+\alpha)(|x|+1).$$

Thus, the SDE (3) has an unique strong non-exploding solution (see [5, Chapter III,2])). Now, we consider the diffusion $\eta_t := \theta_t - 1$. It is a diffusion with drift and diffusion coefficients equal to $\tilde{b}(y) = (\alpha+1)|y+1|$ and $\tilde{\sigma}(y) = \sqrt{|y^2+2y|}$, respectively. We observe that the point 0 is an isolated singular point for η. For $\alpha \geq 0$ and $a > 0$ we have

$$\int_0^a \exp\Big(\int_x^a \frac{2\tilde{b}(y)}{\tilde{\sigma}^2(y)} dy\Big) dx = \int_0^a \Big(\frac{a^2+2a}{x^2+2x}\Big)^{\alpha+1} dx \geq a^{\alpha+1} \int_0^a x^{-(\alpha+1)} dx = \infty.$$

Therefore, from Theorems [8, Theorem 2.16] and [8, Theorem 2.17] follows that $\eta_t > 0$ for all $t > 0$, provided $\eta_0 > 0$. So, if $\theta_0 > 1$, then $\theta_t > 1$ for all $t > 0$. Hence, the process θ satisfies for $\alpha \geq 0$ the SDE

$$d\theta_t = \sqrt{\theta_t^2 - 1}\, dB_t + (\alpha+1)\theta_t dt. \tag{4}$$

It turns out, that in the case of $\alpha \geq 0$ we can recover a hyperbolic Bessel process from the process θ. As usual, by $ar\cosh$ we denote the inverse function of cosh.

Theorem 2.1. *If $\alpha \geq 0$ and $x > 1$ the process $R_t = ar\cosh(\theta_t)$ is a hyperbolic Bessel process, where θ is the solution of* (4).

Proof. Observe that, by definition,

$$R_t = ar\cosh(\theta_t) = \ln\Big(\theta_t + \sqrt{\theta_t^2 - 1}\Big) > 0. \tag{5}$$

As $\theta_t > 1$ for all $t \geq 0$, we can use the Itô lemma to obtain that R satisfies (1). □

Proposition 2.2. *A hyperbolic Bessel process R with index $\alpha \geq -1$ does not explode.*

Proof. For $\alpha \geq 0$ we use Theorem 2.1, which implies that R does not explode since θ does not explode. For $\alpha = -\frac{1}{2}$ the process R is Brownian motion. For $\alpha \in [-1;0) \setminus \{-\frac{1}{2}\}$, we observe that $\cosh(R)$ and θ have the same generator $\mathcal{A}_\theta$ on C_c^2, the space of twice continuously differentiable functions with compact support. From the uniqueness of solution of martingale problem induced by $\mathcal{A}_\theta$ we conclude

that $\cosh(R)$ and θ have the same distributions. The diffusion θ does not explode, and consequently R does not explode. □

Now we define a process $\Gamma^{(\alpha)}$, which appears in the sequel. Let B be a standard Brownian motion, $\alpha \in \mathbb{R}$, $\lambda > 0$ and

$$\Gamma_t^{(\alpha)} = \frac{e^{B_t+\alpha t}}{1+\lambda\int_0^t e^{B_u+\alpha u}du}. \tag{6}$$

There are connections between processes $\Gamma^{(-\frac{1}{2})}$ and cosh of a hyperbolic Bessel process with $\alpha = -1$. Namely, in Jakubowski and Wiśniewolski [12, Theorem 2.4] it is proved that for a hyperbolic Bessel process R with index $\alpha = -1$, $R_0 = x \geq 0$, and $\lambda \geq 0$ we have

$$\mathbb{E}e^{-\lambda\cosh(R_t)} = e^{-\lambda}\mathbb{E}e^{-\lambda(\cosh(x)-1)\Gamma_t^{(-\frac{1}{2})}}. \tag{7}$$

Now we present a form of Laplace transform of cosh of hyperbolic Bessel processes with $\alpha \geq -1/2$, one of the most important results of this paper.

Theorem 2.3. *Let $\alpha \geq -1/2$ and $t > 0$ be fixed. If R is a hyperbolic Bessel process of the form* (1) *with $x \geq 0$, then for $\lambda > 0$*

$$\mathbb{E}\Big[\exp\Big(-\lambda\cosh R_t\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\lambda\cosh(x)V_t - \frac{\lambda^2}{2}\int_0^t V_u^2du\Big)\Big], \tag{8}$$

where $V_t = e^{\left(\alpha+\frac{1}{2}\right)t+B_t}$ and B is a standard Brownian motion.

Proof. Set $\theta_t = \cosh R_t > 0$. So, $\theta_0 = \cosh x$ and

$$d\theta_t = \sqrt{\theta_t^2-1}\;dB_t + a\theta_t dt, \tag{9}$$

where $a = \alpha+1$. From the Itô lemma we obtain

$$de^{-\lambda\theta_t} = -\lambda e^{-\lambda\theta_t}\sqrt{\theta_t^2-1}dB_t - \lambda a e^{-\lambda\theta_t}\theta_t dt + \frac{\lambda^2}{2}e^{-\lambda\theta_t}(\theta_t^2-1)dt. \tag{10}$$

Now, we show that $\int_0^t e^{-\lambda\theta_u}\sqrt{\theta_u^2-1}\;dB_u$ is a martingale. It suffices to prove that

$$\mathbb{E}\int_0^t \theta_u^2 du < \infty \tag{11}$$

for every $t \geq 0$. From (9) we have

$$\begin{aligned}\theta_t^2 &= \Big(\theta_0 + \int_0^t \sqrt{\theta_u^2-1}dB_u + a\int_0^t \theta_u du\Big)^2\\ &\leq 3\Big[\theta_0^2 + \Big(\int_0^t \sqrt{\theta_u^2-1}dB_u\Big)^2 + a^2\Big(\int_0^t \theta_u du\Big)^2\Big].\end{aligned}$$

Using estimation of stochastic integrals and the Jensen inequality for $f(x) = x^2$ we deduce that

$$\mathbb{E}\Big[\Big(\int_0^t \sqrt{\theta_u^2-1}\;dB_u\Big)^2 + a^2\Big(\int_0^t \theta_u du\Big)^2\Big] \leq C\mathbb{E}\int_0^t(\theta_u^2-1)du + a^2t\mathbb{E}\int_0^t \theta_u^2du.$$

Taking $C_1 = 3\theta_0^2$, $C_2 = 3(C+a_1)$ with $a_1 > a^2T$, we conclude that

$$\mathbb{E}\theta_t^2 \leq C_1 + C_2\mathbb{E}\int_0^t \theta_u^2du,$$

for $t \in [0,T]$, so by the Gronwall lemma

$$\mathbb{E}\theta_t^2 \le C_1 e^{C_2 t},$$

and (11) follows.

Therefore, from (10), we infer that

$$\mathbb{E}e^{-\lambda\theta_t} = e^{-\lambda\theta_0} - a\lambda\int_0^t \mathbb{E}[e^{-\lambda\theta_u}\theta_u]du + \frac{\lambda^2}{2}\int_0^t \mathbb{E}[e^{-\lambda\theta_u}(\theta_u^2-1)]du. \tag{12}$$

Define $p(t,\lambda) := \mathbb{E}e^{-\lambda\theta_t}$, so the function p is bounded. Using (12), we deduce that p satisfies the following partial differential equation

$$\frac{\partial p}{\partial t} = -\frac{\lambda^2}{2}p + a\lambda\frac{\partial p}{\partial\lambda} + \frac{\lambda^2}{2}\frac{\partial^2 p}{\partial\lambda^2}, \tag{13}$$

with $p(0,\lambda) = e^{-\lambda\theta_0}$. Set $V_t = V_0 e^{\left(\alpha+\frac{1}{2}\right)t + B_t}$. Then

$$dV_t = V_t dB_t + aV_t dt,$$

with $a = \alpha + 1$. The generator of V is of the form

$$\mathcal{A}_V = \frac{x^2}{2}\frac{d^2}{dx^2} + ax\frac{d}{dx}.$$

Observe, by (12), that $p \in C^{1,2}((0,\infty)\times(0,\infty))$ and from the Feynman-Kac theorem (see [13, Chapter 5 Theorem 7.6]) we know that the only bounded solution of partial differential equation

$$\frac{\partial u}{\partial t} = \mathcal{A}_V u - \frac{x^2}{2}u, \quad u(0,x) = e^{-\theta_0 x},$$

i.e. (13), admits the stochastic representation

$$u(t,x) = \mathbb{E}\exp\Big(-\theta_0 V_t - \frac{1}{2}\int_0^t V_u^2 du\Big),$$

where $V_0 = x$. This implies assertion of the theorem. □

As an easy consequence we obtain

Proposition 2.4. *The Laplace transform of the vector* $(e^{B_t}, \int_0^t e^{2B_u}du)$ *is given by*

$$\mathbb{E}\exp\Big(-\gamma e^{B_t} - \frac{\lambda^2}{2}\int_0^t e^{2B_u}du\Big) = \mathbb{E}\exp\Big(-\lambda\cosh(x+B_t)\Big), \tag{14}$$

for $\gamma > 0$ *and* $\lambda > 0$, *where* $x = ar\cosh\frac{\gamma}{\lambda}$.

Proof. Proposition follows by applying Theorem 2.3 with $\alpha = -\frac{1}{2}$. □

Remark 2.5. The form of the Laplace'a transform of the vector $(e^{B_t}, \int_0^t e^{2B_u}du)$ in the last proposition, i.e. (14), is very simple when it is compared to the form of density of this vector obtained by Matsumoto and Yor [14]. Indeed, the density given in [14] has the oscillating nature in the neighbourhood of 0 and is not convenient for computational use (see [3]). The knowledge of Laplace transform make it possible to invert it numerically and obtain the density of $(e^{B_t}, \int_0^t e^{2B_u}du)$. It is also

important to realize that it enables to obtain the density for $(e^{B_t^{(\mu)}}, \int_0^t e^{2B_u^{(\mu)}}du)$, where $B_t^{(\mu)} := B_t + \mu t$, $\mu \in \mathbb{R}$, as we know from [14] that

$$\mathbb{P}\Big(B_t^{(\mu)} \in dx, \int_0^t e^{2B_u^{(\mu)}}du \in dy\Big) = e^{\mu x - \mu^2 t/2}\mathbb{P}\Big(B_t \in dx, \int_0^t e^{2B_u}du \in dy\Big).$$

Theorem 2.6. *Let $k \in \mathbb{N}$, $\gamma > 0$, $\lambda > 0$ and R be a hyperbolic Bessel process of the form*

$$R_t = ar\cosh(\gamma/\lambda) + B_t + k\int_0^t \coth R_u du. \tag{15}$$

For $\lambda > 0$ we have

$$\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) = (-1)^k e^{\frac{k^2}{2}t}\frac{\partial^k p(\gamma,\lambda)}{\partial\gamma^k},$$

where

$$p(\gamma,\lambda) = \mathbb{E}e^{-\lambda\cosh(ar\cosh(\gamma/\lambda)+B_t)}.$$

Proof. By Proposition 2.4, for $x = ar\cosh(\gamma/\lambda)$,

$$p(\gamma,\lambda) = \mathbb{E}e^{-\lambda\cosh(x+B_t)} = \mathbb{E}\exp\Big(-\gamma e^{B_t} - \frac{\lambda^2}{2}\int_0^t e^{2B_u}du\Big).$$

Define the new probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{kB_t - \frac{k^2}{2}t}.$$

Then $V_t = B_t - kt$ is a standard Brownian motion under $\mathbb{Q}$ and

$$\begin{aligned} p(\gamma,\lambda) &= \mathbb{E}_{\mathbb{Q}}\exp\Big(-kB_t + \frac{k^2}{2}t - \gamma e^{B_t} - \frac{\lambda^2}{2}\int_0^t e^{2B_u}du\Big) \\ &= e^{\frac{k^2}{2}t}\mathbb{E}_{\mathbb{Q}}\exp\Big(-k(V_t+kt) - \gamma e^{(V_t+kt)} - \frac{\lambda^2}{2}\int_0^t e^{2(V_u+ku)}du\Big). \end{aligned}$$

The result follows from Theorem 2.3 after taking k-th derivative of p with respect to γ. □

Proposition 2.7. *For $k \in \mathbb{N}$, $\lambda > 0, \gamma > 0$ and $Y_t = e^{B_t + kt}$*

$$\mathbb{E}\exp\Big(-\gamma Y_t - \frac{\lambda^2}{2}\int_0^t Y_u^2 du\Big) = (-1)^k e^{\frac{k^2}{2}t}\frac{\partial^k p(\gamma,\lambda)}{\partial\gamma^k},$$

where

$$p(\gamma,\lambda) = \mathbb{E}e^{-\lambda\cosh(x+B_t)}$$

and $x = ar\cosh(\frac{\gamma}{\lambda})$.

Proof. Proposition follows from Theorems 2.6 and 2.3. □

Theorem 2.8. *If R is a hyperbolic Bessel process of the form* (1) *with $\alpha \geq -1/2$, then for $\lambda > 0$*

$$\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) = \frac{\sqrt{2\pi}}{\sqrt{t}}e^{-\frac{a^2 t}{2}}\mathbb{E}\Big(1_{\{V_t \geq |B_t|\}}V_t h_t(B_t,x)J_0(\lambda\phi(B_t,V_t))\Big),$$

where $a = \alpha + \frac{1}{2}$, J_0 is the Bessel function of the first kind of order 0, $h_t(z,x) = e^{\frac{z^2}{2t} + az - \lambda\cosh(x)e^z}$, $\phi(x,z) = \sqrt{2}e^{x/2}(\cosh z - \cosh x)^{1/2}$, $z \geq |x|$ and V, B are two independent standard Brownian motions.

Proof. Let $W_t = B_t + at$ and let the new probability measure $\mathbb{Q}$ be given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-aB_t - \frac{a^2}{2}t}.$$

From Theorem 2.3 we have

$$\begin{aligned}\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) &= \mathbb{E}\exp\Big(-\lambda\cosh(x)e^{at+B_t} - \frac{\lambda^2}{2}\int_0^t e^{2(B_u+au)}du\Big)\\ &= \mathbb{E}_{\mathbb{Q}}\exp\Big(aW_t - \frac{a^2}{2}t - \lambda\cosh(x)e^{W_t} - \frac{\lambda^2}{2}\int_0^t e^{2W_u}du\Big)\\ &= e^{-\frac{a^2}{2}t}\mathbb{E}\exp\Big(aB_t - \lambda\cosh(x)e^{B_t} - \frac{\lambda^2}{2}\int_0^t e^{2B_u}du\Big),\end{aligned}$$

so

$$\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) = e^{-\frac{a^2}{2}t}\mathbb{E}\Big[e^{aB_t-\lambda\cosh(x)e^{B_t}}\mathbb{E}\Big(e^{-\frac{\lambda^2}{2}\int_0^t e^{2B_u}du}|B_t\Big)\Big].$$

To finish the proof we use the conditional Laplace transform (see (5.5) in [14])

$$\mathbb{E}\Big(e^{-\frac{\lambda^2}{2}\int_0^t e^{2B_u}du}|B_t = x\Big)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} = \int_{|x|}^{\infty}\frac{z}{\sqrt{2\pi t^3}}e^{-\frac{z^2}{2t}}J_0(\lambda\phi(x,z))dz.$$

□

Theorem 2.9. *If R is a hyperbolic Bessel process of the form* (1) *with $\alpha \geq -1/2$, then for $\lambda > 0$*

$$\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) = e^{-\lambda}\mathbb{E}\Big[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+\frac{1}{2})}}\Big(1+\lambda\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du\Big)^{-\alpha-1}\Big],$$

where B is a standard Brownian motion and $\Gamma_t^{(\alpha)}$ is given by (6).

Proof. Set $Y_t^{(\alpha)} = e^{B_t+\alpha t}$. Define the new measure Q by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-\lambda\int_0^t Y_u^{(\alpha+\frac{1}{2})}dB_u - \frac{\lambda^2}{2}\int_0^t (Y_u^{(\alpha+\frac{1}{2})})^2du}.$$

Theorem 2.3 implies

$$\begin{aligned}\mathbb{E}e^{-\lambda\cosh R_t} &= \mathbb{E}\Big[e^{-\cosh(x)\lambda Y_t^{(\alpha+\frac{1}{2})} - \frac{\lambda^2}{2}\int_0^t (Y_u^{(\alpha+\frac{1}{2})})^2du}\Big]\\ &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-\cosh(x)\lambda Y_t^{(\alpha+\frac{1}{2})} + \lambda\int_0^t Y_u^{(\alpha+\frac{1}{2})}dB_u}\Big]\\ &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-\cosh(x)\lambda Y_t^{(\alpha+\frac{1}{2})} + \lambda(Y_t^{(\alpha+\frac{1}{2})}-1) - (\alpha+1)\lambda\int_0^t Y_u^{(\alpha+\frac{1}{2})}du}\Big] := I,\end{aligned}$$

where we have used the fact that

$$Y_t^{(\alpha+\frac{1}{2})} = 1 + \int_0^t Y_u^{(\alpha+\frac{1}{2})}dB_u + (\alpha+1)\int_0^t Y_u^{(\alpha+\frac{1}{2})}du. \tag{16}$$

From the Girsanov theorem, the process $V_t = B_t + \lambda\int_0^t Y_u^{(\alpha+\frac{1}{2})}du$ is a standard Brownian motion under $\mathbb{Q}$. By the result of Alili, Matsumoto, and Shiraishi [2, Lemma 3.1], we have

$$B_t = V_t - \ln\Big(1+\lambda\int_0^t e^{V_u+(\alpha+\frac{1}{2})u}du\Big). \tag{17}$$

Thus

$$Y_t^{(\alpha+\frac{1}{2})} = \frac{e^{V_t+(\alpha+\frac{1}{2})t}}{1+\lambda\int_0^t e^{V_u+(\alpha+\frac{1}{2})u}du},$$

and

$$\begin{aligned} I &= e^{-\lambda}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\lambda(\cosh(x)-1)\frac{e^{V_t+(\alpha+\frac{1}{2})t}}{1+\lambda\int_0^t e^{V_u+(\alpha+\frac{1}{2})u}du}-(\alpha+1)(V_t-B_t)}\Big] \\ &= e^{-\lambda}\mathbb{E}\Big[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+\frac{1}{2})}}\Big(1+\lambda\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du\Big)^{-\alpha-1}\Big]. \end{aligned}$$

□

Considering $\alpha=-1/2$ in the previous result we obtain

Corollary 2.10. *For $\lambda>0$, $x\geq 0$,*

$$\mathbb{E}\Big[\exp\Big(-\lambda\cosh(B_t+x)\Big)\Big] = e^{-\lambda}\mathbb{E}\Big[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(0)}}\Big(1+\lambda\int_0^t e^{B_u}du\Big)^{-1/2}\Big].$$

In the sequel we use the following notation:

$$A_t^{(\alpha)} = \int_0^t e^{2(B_u+\alpha u)}du, \quad \alpha\in\mathbb{R}, \qquad A_t = A_t^{(0)} \tag{18}$$

with B being a standard Brownian motion.

Theorem 2.11. *If R is a hyperbolic Bessel process of the form* (1) *with $\alpha=0$ and $x=0$, then the density of $\cosh R_t$ on $[1,\infty)$, $t>0$, has the form*

$$\mathbb{P}(\cosh R_t\in dz) = \mathbb{E}\Big[\frac{1}{4A_{\frac{t}{4}}^{(1)}}\exp\Big(-\frac{z-1}{4A_{\frac{t}{4}}^{(1)}}\Big)\Big]dz. \tag{19}$$

Proof. Fix $t>0$. Using Theorem 2.9 for $\alpha=x=0$ and $\lambda>0$ we obtain

$$\mathbb{E}e^{-\lambda(\cosh R_t-1)} = \mathbb{E}\Big[\frac{1}{1+\lambda\int_0^t e^{B_u+\frac{u}{2}}du}\Big].$$

From the last equality and the scaling property of Brownian motion we have

$$\mathbb{E}e^{-\lambda(\cosh R_{4t}-1)} = \mathbb{E}\Big[\frac{1}{1+4\lambda\int_0^t e^{B_{4u}+2u}du}\Big] = \mathbb{E}\Big[\frac{1}{1+4\lambda A_t^{(1)}}\Big]. \tag{20}$$

From Theorem 2.8 in [12] we know that

$$\mathbb{E}\Big[\frac{1}{1+4\lambda A_t^{(1)}}\Big] = 1-4\lambda\int_0^\infty G_t(y)e^{-4\lambda y}dy,$$

where $G_t(y)=\mathbb{E}\Big(e^{-\frac{y}{A_t^{(1)}}}\Big)$.

Since G_t is differentiable, integration by parts formula yields

$$\mathbb{E}\Big[\frac{1}{1+4\lambda A_t^{(1)}}\Big] = 1+\int_0^\infty G_t(y)(e^{-4\lambda y})'dy = -\int_0^\infty G_t'(y)e^{-4\lambda y}dy.$$

From the last equality and the definition of G_t we have

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{1+4\lambda A_t^{(1)}}\Big] &= \int_0^\infty e^{-4\lambda y}\mathbb{E}\Big[\frac{1}{A_t^{(1)}}\exp\Big(-\frac{y}{A_t^{(1)}}\Big)\Big]dy \\ &= \int_0^\infty e^{-\lambda z}\mathbb{E}\Big[\frac{1}{4A_t^{(1)}}\exp\Big(-\frac{z}{4A_t^{(1)}}\Big)\Big]dz. \qquad (21)\end{aligned}$$

From (20) and (21) we conclude that

$$\mathbb{P}\Big((\cosh R_{4t}-1)\in dz\Big) = \mathbb{E}\Big[\frac{1}{4A_t^{(1)}}\exp\Big(-\frac{z}{4A_t^{(1)}}\Big)\Big]dz,$$

for $z \geq 0$, which ends the proof. □

Remark 2.12. Using the explicit form of the density of $A_t^{(1)}$ and Theorem 2.11, we can obtain the integral form of density of $\cosh R_t$, with $\alpha = 0$ and $x = 0$.

Proposition 2.13. *If R is a hyperbolic Bessel process of the form* (1) *with $\alpha = 0$ and $x = 0$ then on $[1, \infty)$*

$$\mathbb{P}(\cosh R_t \in dz) = -\frac{1}{4}G'_{\frac{t}{4}}\Big(\frac{z-1}{4}\Big)dz, \qquad (22)$$

for $t > 0$, where

$$G_t(y) = e^{-t/2}\mathbb{E}\exp\Big(B_t + \frac{1}{2t}\Big(B_t^2 - \varphi_y^2(B_t)\Big)\Big), \qquad (23)$$

B is a standard Brownian motion and

$$\varphi_y(z) = \ln\Big(ye^{-z} + \cosh(z) + \sqrt{y^2e^{-2z} + \sinh^2(z) + 2ye^{-z}\cosh(z)}\Big). \qquad (24)$$

Proof. As before, let $G_t(y) = \mathbb{E}\Big(e^{-\frac{y}{A_t^{(1)}}}\Big)$. As an easy consequence of the Matsumoto-Yor result [14, Theorem 5.6] we obtain that G_t is given by (23) (see the proof of Theorem 2.5 in [12]). Proposition follows from Theorem 2.11 and observation that

$$\mathbb{E}\Big[\frac{1}{4A_{\frac{t}{4}}^{(1)}}\exp\Big(-\frac{z-1}{4A_{\frac{t}{4}}^{(1)}}\Big)\Big] = -\frac{1}{4}\,G'_{\frac{t}{4}}\Big(\frac{z-1}{4}\Big).$$

□

In the next theorem we prove that $R_t \overset{\text{(law)}}{=} F(B, A, X)$ for some functional F, so the distribution of R_t can be presented as functional of A, B and a squared Bessel process X of index α independent of B.

Theorem 2.14. *If R is a hyperbolic Bessel process of the form* (1) *with $\alpha \geq -1/2$, $R_0 = x$, then*

$$\cosh(R_t) \overset{\text{(law)}}{=} 1 + 2A_{t/4}^{(2\alpha+1)}X_1 \qquad (25)$$

for every $t > 0$, where X is a squared Bessel process of index α, independent of a standard Brownian motion B and starting from $\frac{1}{2}(\cosh(x)-1)e^{2B_{t/4}^{(2\alpha+1)}}/A_{t/4}^{(2\alpha+1)}$. Moreover, for $\alpha > -1/2$,

$$R_t \overset{\text{(law)}}{=} ar\cosh\Big(1 + 2A_{t/4}^{(2\alpha+1)}X_1\Big). \qquad (26)$$

Proof. From Theorem 2.9 and the form of Laplace transform of squared Bessel process (see [17, Chapter XI, p.441]) we have, for arbitrary $\lambda > 0$,

$$\begin{aligned}\mathbb{E}\exp\Big(-\lambda\cosh R_t\Big) &= e^{-\lambda}\mathbb{E}\Big[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+\frac{1}{2})}}\Big(1+\lambda\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du\Big)^{-\alpha-1}\Big]\\ &= e^{-\lambda}\mathbb{E}\exp\Big(-\lambda e^{B_t+(\alpha+\frac{1}{2})t}\hat{X}_{\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du/(2e^{B_t+(\alpha+\frac{1}{2})t})}\Big),\end{aligned}$$

where $\hat{X}$ is a squared Bessel process of index α, independent of a standard Brownian motion B and starting from $(\cosh(x)-1)$. It is now clear that

$$\cosh(R_t) \overset{(\text{law})}{=} 1 + e^{B_t+(\alpha+\frac{1}{2})t}\hat{X}_{\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du/(2e^{B_t+(\alpha+\frac{1}{2})t})}. \tag{27}$$

Using the scaling property of squared Bessel process (see [17, Chapter XI, Prop. 1.6]) we obtain

$$1 + e^{B_t+(\alpha+\frac{1}{2})t}\hat{X}_{\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du/(2e^{B_t+(\alpha+\frac{1}{2})t})} = 1 + \frac{1}{2}\overline{X}_1\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du, \tag{28}$$

where $\overline{X}$ is a squared Bessel process of index α independent of B and starting from the stochastic point $\overline{X}_0 = 2(\cosh(x)-1)e^{B_t+(\alpha+\frac{1}{2})t}/\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du$. Hence, taking $4t$ instead of t, we infer that

$$\cosh(R_{4t}) \overset{(\text{law})}{=} 1 + \frac{1}{2}\overline{X}_1\int_0^{4t} e^{B_u+(\alpha+\frac{1}{2})u}du \overset{(\text{law})}{=} 1 + 2\overline{A}_t^{(2\alpha+1)}X_1,$$

where X is a squared Bessel process of index α independent of a standard Brownian motion $\overline{B}$ and starting from $X_0 = \frac{1}{2}(\cosh(x)-1)e^{2\overline{B}_t^{(2\alpha+1)}}/\overline{A}_t^{(2\alpha+1)}$, where $\overline{A}_t^{(2\alpha+1)}$ is given by (18) with $\overline{B}$ instead od B. The proof is complete. □

Remark 2.15. Fix $\alpha > -1/2$. For every $t > 0$, from the proof of Theorem 2.14 we obtain that

$$R_t \overset{(\text{law})}{=} ar\cosh\Big(1 + (1/2)a_tX_1\Big), \tag{29}$$

where $a_t = \int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du$, X is a squared Bessel process of index $\alpha > -1/2$ independent of a standard Brownian motion B and starting from the random point $X_0 = 2(\cosh(x)-1)e^{B_t+(\alpha+\frac{1}{2})t}/\int_0^t e^{B_u+(\alpha+\frac{1}{2})u}du$ (see (27) and (28)).

The last theorem, in the special case $x = 0$, gives

Proposition 2.16. *If $x = 0$ and $\alpha \geq -1/2$, then for every t the density function of $\cosh(R_t)$ on $[1,\infty)$ has the form*

$$\mathbb{P}(\cosh(R_t) \in dz) = \frac{1}{4^{\alpha+1}}\frac{(z-1)^\alpha}{\Gamma(\alpha+1)}\mathbb{E}\Big[e^{-(z-1)/(4A_{t/4}^{(2\alpha+1)})}\frac{1}{\Big(A_{t/4}^{(2\alpha+1)}\Big)^{\alpha+1}}\Big]dz. \tag{30}$$

If $x = 0$ and $\alpha > -1/2$, then for every t the density function of R_t on $[0,\infty)$ has the form

$$\mathbb{P}(R_t \in dz) = \frac{1}{4^{\alpha+1}}\frac{(\cosh(z)-1)^\alpha\sinh(z)}{\Gamma(\alpha+1)}\mathbb{E}\Big[e^{-(\cosh(z)-1)/(4A_{t/4}^{(2\alpha+1)})}\frac{1}{\Big(A_{t/4}^{(2\alpha+1)}\Big)^{\alpha+1}}\Big]dz. \tag{31}$$

Proof. From Theorem 2.14 for $x = 0$ and $\alpha \geq -1/2$ we obtain

$$\cosh(R_t) \overset{\text{(law)}}{=} 1 + 2A^{(2\alpha+1)}_{t/4} X_1, \tag{32}$$

where X is a squared Bessel process of index α, independent of B and starting from 0. For $\alpha > -1/2$ it holds that

$$R_t \overset{\text{(law)}}{=} ar\cosh\Big(1 + 2A^{(2\alpha+1)}_{t/4} X_1\Big). \tag{33}$$

Hence for $x = 0$ and $\alpha \geq -1/2$

$$\mathbb{P}(\cosh(R_t) \leq z) = \mathbb{P}\Big(X_1 \leq \frac{z-1}{2A^{(2\alpha+1)}_{t/4}}\Big)$$

and for $x = 0$ and $\alpha > -1/2$

$$\mathbb{P}(R_t \leq z) = \mathbb{P}\Big(X_1 \leq \frac{\cosh(z)-1}{2A^{(2\alpha+1)}_{t/4}}\Big).$$

From the last equality and the form of the density of squared Bessel process (see [17, Chapter XI, Cor. 1.4]) we have for $x = 0$ and $\alpha \geq -1/2$

$$\mathbb{P}(\cosh(R_t) \in dz) = \Big(\frac{1}{4}\Big)^{\alpha+1} \frac{(z-1)^\alpha}{\Gamma(\alpha+1)} \mathbb{E}\Big[e^{-(z-1)/(4A^{(2\alpha+1)}_{t/4})} \frac{1}{\Big(A^{(2\alpha+1)}_{t/4}\Big)^{\alpha+1}}\Big] dz.$$

and for $x = 0$ and $\alpha > -1/2$

$$\mathbb{P}(R_t \in dz) =$$
$$= \Big(\frac{1}{4}\Big)^{\alpha+1} \frac{(\cosh(z)-1)^\alpha \sinh(z)}{\Gamma(\alpha+1)} \mathbb{E}\Big[e^{-(\cosh(z)-1)/A^{(2\alpha+1)}_{t/4}} \frac{1}{\Big(A^{(2\alpha+1)}_{t/4}\Big)^{\alpha+1}}\Big] dz.$$

□

Remark 2.17. Using the explicit form of the density of $A^{(2\alpha+1)}_t$ and Proposition 2.16, we can obtain the integral form of density of R_t for $\alpha \geq -1/2$ and $R_0 = 0$. This formula is new and differs significantly from the density obtained by Borodin (see formula (5.4) in [4]).

The next two facts follows immediately from 2.14 and in slightly different forms can be found in [14].

Corollary 2.18. *For any $z \geq 1$ we have*

$$\mathbb{E}\Big[\frac{1}{\sqrt{A_t}} \exp\Big(-\frac{(\cosh(z)-1)}{4A_t}\Big)\Big] = \sqrt{\frac{2}{t}} \frac{\sqrt{\cosh(z)-1}}{\sinh(z)} \ \exp\Big(-\frac{z^2}{8t}\Big).$$

Proof. It follows from Proposition 2.16 for $x = 0$ and $\alpha = -\frac{1}{2}$ and the form of density of $\cosh(B_t)$. □

Proposition 2.19. *For every $\lambda > 0$ we have*

$$\mathbb{E}A^\lambda_t = \frac{\mathbb{E}(\cosh(B_{4t}) - 1)^\lambda}{2^\lambda \mathbb{E}(B_1)^\lambda},$$

where B is a standard Brownian motion.

Proof. It follows from Theorem 2.14 for $x = 0$ and $\alpha = -1/2$. □

In the next proposition we present an alternative purely probabilistic proof of generalization of Bougerol's identity (see [1], [2, Prop. 6.1] or [14, Section 3]).

Proposition 2.20. *Let B and W be two independent standard Brownian motions.*
a) [Bougerol's identity] For $t > 0$,

$$\sinh(B_t) \overset{\text{(law)}}{=} W_{A_t}. \tag{34}$$

b) If $x \in \mathbb{R}$, then

$$\sinh(x + B_t) \overset{\text{(law)}}{=} \sinh(x)e^{B_t} + W_{A_t}. \tag{35}$$

Proof. a)

$$(W_{A_t})^2 \overset{\text{(law)}}{=} A_t(W_1)^2 \overset{\text{(law)}}{=} \frac{\cosh(B_{4t}) - 1}{2} = \frac{e^{2\frac{1}{2}B_{4t}} + e^{-2\frac{1}{2}B_{4t}} - 2}{4} \overset{\text{(law)}}{=} (\sinh(B_t))^2,$$

where in the second equality we use Theorem 2.14 with $x = 0$ and $\alpha = -1/2$. Hence, using the fact that W_{A_t} and $\sinh(B_t)$ are symmetric random variable we obtain (34).
b) By the same arguments as in a), we conclude that

$$|\sinh(x + B_t)| \overset{\text{(law)}}{=} |\sinh(x)e^{B_t} + W_{A_t}|$$

and we can expect that we can omit modulus. Indeed, define

$$M_t = e^{B_t}\big(\sinh(x) + \int_0^t e^{-B_u} dW_u\big).$$

Observe, by Proposition 2.1 in [9], that

$$\sinh(x)e^{B_t} + W_{A_t} \overset{\text{(law)}}{=} \sinh(x)e^{B_t} + e^{B_t}\int_0^t e^{-B_u} dW_u = M_t.$$

It is easy to check that M is a diffusion with the generator

$$\mathcal{A}_M = \frac{1}{2}(x^2 + 1)\frac{d^2}{dx^2} + \frac{1}{2}x\frac{d}{dx}. \tag{36}$$

It is also evident that SDE corresponding to the above generator has a unique strong solution. To finish the proof it is enough to observe that $\widehat{M}_t := \sinh(x + B_t)$ is a diffusion such that generators of M and $\widehat{M}$ are equal on C_c^2, so $M \overset{\text{(law)}}{=} \widehat{M}$. □

In the next proposition, we deduce from Theorem 2.20 the simple form of characteristic function of vector (e^{B_t}, W_{A_t}).

Proposition 2.21. *Let $u \in \mathbb{R}$. Then, for $v \neq 0$,*

$$\mathbb{E}e^{iue^{B_t} + ivW_{A_t}} = \mathbb{E}e^{iv\sinh(ar\sinh(u/v) + B_t)}. \tag{37}$$

Proof. From Theorem 2.20 we infer that

$$\frac{u}{v}e^{B_t} + W_{A_t} \overset{\text{(law)}}{=} \sinh(ar\sinh(u/v) + B_t),$$

which implies (37). □

Now we establish the representation of law of the process $\sinh(R)$, where R is a hyperbolic Bessel process with index $\alpha = 0$, in terms of functionals of independent Brownian motions.

Theorem 2.22. *Let $x \geq 0$ and R be hyperbolic Bessel process starting from x and with the index $\alpha = 0$. Then*

$$(\sinh(R_t), t \geq 0) \stackrel{(\mathrm{law})}{=}$$

$$\Big(e^{-B_t+t/2}\Big(\sinh(x) + \int_0^t e^{B_u-u/2}dV_u\Big)^2 + \Big(\int_0^t e^{B_u-u/2}dZ_u\Big)^2\Big)^{\frac{1}{2}}, t \geq 0\Big), \tag{38}$$

where B, V, Z are three independent standard Brownian motions.

Proof. Define $\xi_t = \sinh^2(R_t)$. Then the diffusion ξ satisfies the following SDE

$$d\xi_t = 2\sqrt{\xi_t^2 + \xi_t}dB_t + (2 + 3\xi_t)dt,$$

and the generator of ξ is of the form

$$\mathcal{A}_\xi = 2(x^2 + x)\frac{d^2}{dx^2} + (2 + 3x)\frac{d}{dx}. \tag{39}$$

Let's denote the diffusion and drift coefficients of ξ by $\sigma(x) = 2\sqrt{x^2 + x}$ and $\mu(x) = 2 + 3x$. Observe that for any $x \geq 0$ and $y \in (x - 1, x + 1)$

$$\begin{aligned}
\Big(\sigma(x) - \sigma(y)\Big)^2 &= 4\Big(\sqrt{x^2 + x} - \sqrt{y^2 + y}\Big)^2 \leq 4\Big(|x^2 - y^2| + |x - y|\Big) \\
&= 4|x - y|(|x + y| + 1) \leq 4|x - y|(2|x| + 2) \\
&\leq 4|x - y|(\sigma(x) + 2) \leq 4|x - y|(\sigma^2(x)/2 + 5/2).
\end{aligned}$$

The uniqueness of the solution for the corresponding SDE follows now from [17, Chapter IX Ex. 3.14]. Thus, there is a unique solution of a martingale problem induced by $\mathcal{A}_\xi$.

For a standard Brownian motion B we define, as previously, $Y_t = e^{B_t - t/2}$. We consider now a SDE of the form

$$dX_t = 2\sqrt{X_t}Y_t dW_t + 2Y_t^2 dt, \tag{40}$$

where $X_0 = \sinh^2(x)$ and W is a standard Brownian motion independent of B. Observe, using the Itô lemma, that the process $\psi = \frac{X}{Y^2}$ is a diffusion with the generator $\mathcal{A}_\psi$ having the same form as $\mathcal{A}_\xi$ on on C_c^2. Hence, by uniqueness of solution of the martingale problem induced by $\mathcal{A}_\xi$, we obtain that the processes $|\sinh(R)|$ and $|\sqrt{X}/Y|$ have the same law. To skip the absolute value we realize that both processes are nonnegative. Moreover, the SDE (40) has a unique weak solution. Indeed, using the change of time $\tau_t = \inf\{s \geq 0 : \int_0^s Y_u^2 du \geq t\}$ we obtain that the process $X_{\tau_\cdot}$ is a square of 2-dimensional Bessel process, and in consequence it is not difficult to see that the process X inherits "good" properties of a square of 2-dimensional Bessel process. To finish the proof we observe that the unique weak solution of (40) can be written as

$$X_t = (\sinh(x) + \int_0^t Y_u dV_u)^2 + (\int_0^t Y_u dZ_u)^2. \tag{41}$$

To verify that (41) satisfies (40) it is enough to realize that W defined as

$$W_t := \int_0^t 1_{\{\Lambda_s \neq 0\}} \frac{(\sinh(x) + \int_0^s Y_u dV_u)dV_s + (\int_0^s Y_u dZ_u)dZ_s}{\sqrt{(\sinh(x) + \int_0^s Y_u dV_u)^2 + (\int_0^s Y_u dZ_u)^2}}, \tag{42}$$

where $\Lambda_s = (\sinh(x) + \int_0^s Y_u dV_u)^2 + (\int_0^s Y_u dZ_u)^2$, is a standard Brownian motion. Simple use of the Itô lemma finishes the proof. □

It turns out that the methods of the proof of Theorem 2.22 allows also to find a representation of law of hyperbolic Bessel process with index $\alpha > -1/2$.

Theorem 2.23. *Let $x \geq 0$ and R be hyperbolic Bessel process starting from x with the index $\alpha > -1/2$. Then*

$$\Big(\sinh(R_t), t \geq 0\Big) \overset{\text{(law)}}{=} \Big(\sqrt{X_t}/Y_t, t \geq 0\Big), \tag{43}$$

where $Y_t = e^{B_t - (\alpha+1/2)t}$, X_t satisfies the SDE

$$dX_t = 2\sqrt{X_t} Y_t dW_t + 2(1+\alpha) Y_t^2 dt, \tag{44}$$

$X_0 = \sinh^2(x)$ and B, W are two independent standard Brownian motions.

Proof. The proof goes in the same way as the proof of previous theorem. □

Theorem 2.24. *Let X and Y be as in Theorem 2.23. For any $w \geq 0, t \geq 0$*

$$\mathbb{P}\Big(\frac{\sqrt{X_t}}{Y_t} \leq w\Big) = \mathbb{E}\Big[F_S\Big(\int_0^t Y_u^2 du, wY_t\Big)\Big], \tag{45}$$

where $F_S(t,x) = \mathbb{P}(S_t \leq w)$ and S is a Bessel process of dimension $2(1+\alpha)$ with $S_0 = \sinh(x)$.

Proof. Observe that the process X_{τ_t} is a square of $2(1+\alpha)$-dimensional Bessel process for $\tau_t = \inf\{s \geq 0 : \int_0^s Y_u^2 du \geq t\}$. Observe also that conditionally, under the knowledge of trajectory of $(Y_s, s \leq t)$, the process X_{τ_t} is still a square of $2(1+\alpha)$-dimensional Bessel process. Hence

$$\begin{aligned}\mathbb{P}\Big(\frac{\sqrt{X_t}}{Y_t} \leq w\Big) &= \mathbb{P}\Big(S_{\int_0^t Y_u^2 du} \leq wY_t\Big) = \mathbb{E}\Big[\mathbb{P}\Big(S_{\int_0^t Y_u^2 du} \leq wY_t \Big| \sigma(Y_u, u \leq t)\Big)\Big] \\ &= \mathbb{E}\Big[F_S\Big(\int_0^t Y_u^2 du, wY_t\Big)\Big].\end{aligned}$$

□

Proposition 2.25. *Let $k \in \mathbb{N}$, $x \geq 0$ and Y be a given continuous process such that $\mathbb{E}\int_0^t Y_u^2 du < \infty$ for any $t \geq 0$. Then the unique solution of the SDE*

$$dX_t = 2\sqrt{X_t} Y_t dW_t + kY_t^2 dt, \tag{46}$$

where W is a standard Brownian motion independent of Y and $X_0 = x$, is of the form

$$X_t = \Big(\sqrt{x} + \int_0^t Y_u dW_u^1\Big)^2 + \sum_{i=2}^k \Big(\int_0^t Y_u dW_u^i\Big)^2, \tag{47}$$

where W^i, $i = 1, \ldots, k$, are independent standard Brownian motions.

Proof. As previously, uniqueness of solution of (46) follows from the fact that the standard change of time ($\tau_t = \inf\{u : \int_0^u Y_s^2 ds > t\}$) results in X becoming a square of k-dimensional Bessel process. We complete the proof by the direct checking that X defined by (47) satisfies (46) using analogous arguments as in the proof of Theorem 2.24. □

2.1. **Hyperbolic Bessel processes with stochastic time.**

Proposition 2.26. *Let T_δ be a random variable with exponential distribution with the parameter $\delta > 0$. Assume that T_λ is independent of a standard Brownian motion B. Then for a hyperbolic Bessel process of the form* (1) *with $\alpha \geq -1/2$, we have*

$$\mathbb{E} \exp\Big(- \lambda \cosh R_{T_\delta} \Big) = \int_0^\infty \int_0^\infty e^{-\lambda u \cosh(x) - \frac{1}{2} v} p^\gamma(u, 1, y) dy du, \tag{48}$$

where for $\gamma = \sqrt{2\delta + (\alpha + \frac{1}{2})^2}$,

$$p^\gamma(u, 1, y) = \frac{\delta}{y^{\frac{1}{2} - \alpha} u} e^{-(y^2+1)/(2u)} I_\gamma\Big(\frac{y}{u}\Big) \tag{49}$$

and I_γ is modified Bessel function.

Proof. We use Theorem 2.3 and the result of Matsumoto-Yor [14, Theorem 4.11]. □

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